

The Frieden-Soffer Extreme Physical Information Principle in a Non-extensive Setting

L. P. Chimento^{2,3}, F. Pennini¹ and A. Plastino^{1,31}

¹*Physics Department, National University La Plata,
C. C. 727, 1900 La Plata, Argentina*

²*Departamento de Física, Facultad de Ciencias
Exactas y Naturales, Universidad de Buenos Aires, Ciudad
Universitaria, Pabellón I, 1428 Buenos Aires, Argentina*

³*Argentine National Research Council (CONICET)*

We show that the Frieden and Soffer's Extreme Physical Information principle, applied to a non-extensive *statistical* scenario, yields solutions to several well-known classical *dynamical* problems.

I. INTRODUCTION

A. Fisher's Information for Translation Families

A very active area of theoretical endeavour nowadays is that concerned with the investigation of properties and applications of Fisher's information measure for translation families (to be denoted herefrom as I), whose applications to diverse problems in theoretical physics have been pioneered by Frieden, Soffer, Nikolov, Silver, and others [1–13]. They have unveiled many properties of Fisher's information measure and clarified its relation to Shannon's logarithmic information measure

$$S = - \int dx f(x) \ln f(x), \quad (1)$$

where $f(x)$ is, of course, a probability density (for the sake of simplicity we restrict our considerations here to the one dimensional case).

A word of caution: the information I that we are here considering is that *particular version of the one originally introduced by Fisher* [1] that applies for translation families, i.e., refers to a measure of the uncertainty in determining a *position parameter* x by a maximum likelihood estimation [11]. It is seen [1] that the best possible estimator of x , after a large number of samples is examined, suffers a mean-square deviation e^2 from $\langle x \rangle$ that minimizes the so-called Cramer-Rao bound [1–5]

$$I e^2 \geq 1, \quad (2)$$

where Fisher's information I for translation families reads

$$I = \int dx f(x) \left\{ \frac{df}{dx} \right\}^2, \quad (3)$$

i.e.,

$$I = \langle \left\{ \frac{df}{dx} \right\}^2 \rangle. \quad (4)$$

B. Non-extensive Thermostatistics

Another area of current interest, for a variety of physical reasons, is that related to nonlinear formalisms. Among them we can single out nonextensive thermostatistics (NET) [14], characterized by a (real) parameter q . NET refers, of course, to the use of Tsallis information measure (see Eq. (6) below) for a general value of the parameter q .

Much work in this respect has been performed recently (165 refereed papers at the time of this writing). NET has been proved capable of overcoming the inability of conventional Statistical Mechanics (for which one has $q = 1$) to address problems in some fields. We may cite, among such NET successful applications, those to astrophysical

problems [15, 16], to Lévy flights [17], to turbulence phenomena [18], to simulated annealing [19], etc. The interested reader is referred to [14] for additional references.

Within a NET context, if we associate a probability distribution $\{p_i\}$ to a set of W different microstates i

$$\sum_i^W p_i = 1, \quad (5)$$

the concomitant Tsallis information measure reads [14]

$$S_q = (q - 1)^{-1} \sum_i^W p_i [1 - p_i^{q-1}]. \quad (6)$$

It is easy to see that, for $q = 1$, one regains the conventional Boltzmann-Gibbs-Shannon logarithmic form [14]. Otherwise, we are led into the realm of nonextensivity [14, 15, 20–22]. Indeed, in order to study the limit $q \rightarrow 1$ we rewrite Eq. (6) in the fashion

$$S_q = \sum_i p_i \left\{ \frac{1 - e^{(q-1) \ln p_i}}{q - 1} \right\}, \quad (7)$$

and find that for $q \rightarrow 1$

$$S_1 \equiv \lim_{q \rightarrow 1} S_q = -k \sum_i p_i \ln p_i, \quad (8)$$

i.e., for $q = 1$ Tsallis' entropy coincides with the Gibbs-Shannon one.

C. Entropic and Fisher's measures within a NET context

Within a NET context one identifies the information measure, of course, with the Tsallis generalized entropy (6). It goes without saying that extension of Fisher's ideas to a NET scenario should be of interest. In this respect, some preliminary work has been advanced in [23]. In that paper, Fisher's information measure I is generalized to a nonextensive environment (yielding a suitable quantity I_q).

D. The goal of the present communication

Here we will show that generalization of the most important physical application of Fisher's ideas, the so-called Frieden-Soffer principle of Extreme Physical Information [7] (to a non-extensive setting) yields intriguing results concerning several important equations of classical dynamics. The concomitant analysis will also illuminate interesting aspects of the different NET statistics that arise as one varies the all important Tsallis' q -parameter.

As a first step in such a direction, and in order to establish a convenient notation, we begin by refreshing, in Section II: i) the extension of Fisher-Frieden ideas to non-extensive settings, and ii) the Frieden-Soffer Principle concerning extremization of Physical Information [7]. In Section III we discuss the application of this Principle in a non-extensive environment and present the rather surprising results of the present Communication in Section IV. Finally, conclusions are drawn in Section V.

II. A REVIEW OF FUNDAMENTAL NOTIONS

A. Fisher's information in a nonextensive setting

A suitable generalization of (4) should read, according to the second NET-postulate [23]

$$I_q = \langle \left\{ \frac{df}{f(x)} \right\}^2 \rangle_q, \quad (9)$$

which is our generalized Fisher's information measure for translation families. In [23] one uses (9) to

- generalize the Cramer-Rao bound [1].
- discuss connections between Tsallis' entropy, on the one hand, and Fisher's measure, on the other one.
- derive a suitably generalized form of the Frieden-Nikolov's bound [24] to the time derivative $\frac{dS_q}{dt}$. This alternative form connects the entropy increase to I_q .

B. The Frieden-Soffer EPI Principle

The Principle of Extreme Physical Information (EPI) is an overall physical theory that is able to unify several sub-disciplines of Physics [7, 12]. In Ref. [7] FS show that the Lagrangians in Physics arise out of a mathematical game between an intelligent observer and Nature (that FS personalize in the appealing figure of a “demon”, reminiscent of the celebrated Maxwell's one). The game's payoff introduces the EPI variational principle, which determines simultaneously the Lagrangian *and* the physical ingredients of the concomitant scenario.

FS [7, 12] envision the following situation: some physical phenomenon is being investigated so as to gather suitable, pertinent data. Measurements must be performed. Any measurement of physical parameters appropriate to the task at hand initiates a relay of information I (Fisher's) from Nature (the demon) into the data. The observer acquires information, in this fashion, that is precisely I . FS assume that this information is (of course!) stored within the system (or inside the demon's mind). The demon's information is called, say, J [7, 12].

Assume now that, due to the measuring process, the system is perturbed, which in turn induces a change δJ of the demon's mind. It is natural to ask ourselves how the data information I will be affected. Enters here FS's EPI: *in its relay from the phenomenon to the data no loss of information should take place*. The ensuing new Conservation Law states that $\delta J = \delta I$, or, rephrasing it

$$\delta(I - J) = 0, \quad (10)$$

so that, defining an information loss (or action) \mathcal{A}

$$\mathcal{A} \equiv I - J, \quad (11)$$

EPI asserts that the whole process described above extremizes \mathcal{A} . FS [7, 12] conclude that the Lagrangian for a given physical environment is not just an *ad-hoc* construct that yields a suitable differential equation. It possesses an intrinsic meaning. Its integral represents the physical information \mathcal{A} for the physical scenario. On such a basis some of the most important equations of Physics can be derived [7, 12].

Within the present context our demon is working in a non-extensive fashion. An appropriate nonextensive information I_q , to be presently introduced, is then required, so that the EPI principle would read

$$\mathcal{A}_q = I_q - J, \quad (12)$$

i.e., using appropriate q -generalizations of the intervening physical quantities. Thus the Frieden-Soffer information transfer game is played here according to

$$\delta(I_q - J) = 0. \quad (13)$$

Information I_q is defined next.

III. THE WORKINGS OF FS'S DEMON IN A NON-EXTENSIVE SCENARIO

A. Introductory remarks

One starts here the FS game by using (13), i.e., by extremizing Fisher's generalized information [23] (the dot indicates differentiation with respect to x)

$$I_q = \int f(x)^{q-2} \dot{f}(x)^2 dx, \quad (14)$$

and considering a physical scenario in which the knowledge of the information demon [7] J is

$$J = \int \tau(f) dx, \quad (15)$$

where the “information density” τ may depend upon q .

As a generalization of the use of probability amplitudes in the $q = 1$ -theory [7, 12], define a new amplitude v obeying

$$f(x) = v(x)^{\frac{2}{q}}. \quad (16)$$

Then the Fisher’s Information for translation families acquires the appearance

$$I_q = \frac{4}{q^2} \int \dot{v}^2 dx. \quad (17)$$

The anzat (16) acquires the Fisher’s form in the special case $q = 1$. Our generalized Lagragian \mathcal{L} reads (in terms of v)

$$\mathcal{L} = \frac{4}{q^2} \dot{v}^2, \quad (18)$$

A general equivalence between the Frieden-Soffer EPI Principle in a non-extensive setting and the Classical Mechanical single-particle Lagragian, can be established by letting the knowledge of the information demon [7] to acquire the form (15). In this case the total Lagrangian density (that includes the demon’s τ) is

$$\mathcal{L}_T = \frac{4}{q^2} \dot{v}^2 - \tau(v). \quad (19)$$

This mechanical analog is completed with the changes

$$x \rightarrow t,$$

and we work in units such that $m = 1$,

$$\textit{statistical amplitude } v \rightarrow \textit{classical dynamical quantity } v,$$

so that one computes the total energy of the particle

$$\mathcal{H} = \frac{\partial \mathcal{L}_T}{\partial \dot{v}} \dot{v} - \mathcal{L}_T, \quad (20)$$

and the energy is given by

$$\frac{4}{q^2} \dot{v}^2 + \tau(v) = E, \quad (21)$$

which is now the new first integral of the motion. The corresponding equation of motion is

$$\ddot{v} + \frac{q^2}{8} \frac{d\tau}{dv} = 0, \quad (22)$$

which illustrates the fact that many classical dynamical problems can be “linked”, via EPI, to a statistical scenario. An effective mass and effective force, that adopt the forms (remember we set above $m = 1$ so that our effective masses become pure numbers)

$$m_{eff} = \frac{8}{q^2}, \quad (23)$$

$$F_{eff} = -\frac{d\tau}{dv}, \quad (24)$$

can be deduced from (22).

In what follows we assume

$$\tau(f) = \lambda f(x), \quad (25)$$

with λ a normalization constant. Notice here an important difference with the scenario devised by Frieden and Soffer. By their approach, information J (and equivalently quantity $\tau(f)$) is solved *for* through the use of a basic property of the measured phenomenon such as unitarity. Depending upon phenomenon, J is often the expectation value of the kinetic energy. By contrast, in Eq. (25) we assume *a form for* $\tau(f)$. Its use is then equivalent to minimizing information I subject to a simple constraint of normalization. This would be a principle of minimum Fisher information [8]. The EPI formalism has a different interpretation of the same situation. Its demon now fixes the phenomenological information J as being merely a statement of normalization. Since any probability distribution function must obey normalization, this represents a minimal amount of information. Hence the demon is being maximally parsimonious in allowing information into the data. The measurer is thereby presented with a scenario of maximum ignorance.

We have

$$\mathcal{L}_T = \mathcal{L} - \lambda f = \frac{4}{q^2} \dot{v}^2 - \lambda v^{\frac{2}{q}}, \quad (26)$$

and are now faced with *the Classical Mechanical Lagrangian for the motion of a particle of effective mass $m_{eff} = 8/q^2$ in an external potential $V(v) = \lambda v^{\frac{2}{q}}$ and such that the effective force is given by $F_{eff} = -\frac{2\lambda}{q} v^{\frac{2}{q}-1}$.*

With

$$\frac{4}{q^2} \dot{v}^2 + \lambda v^{\frac{2}{q}} = E, \quad (27)$$

one obtains a first integral of the motion.

In the v, t variables, the equation of motion is

$$\ddot{v} + \frac{\lambda q}{4} v^{\frac{2-q}{q}} = 0, \quad (28)$$

that, if the real parameter q is set equal to unity (Tsallis' tenets reduce to the orthodox, extensive Boltzmann-Gibbs ones) (27) gives the first integral of the motion for a particle subject to the action of an effective elastic force $F_{eff} = -\lambda v$.

Our first result here can be paraphrased as follows: *the Frieden-Soffer demon is seen to have the ability of translating a statistical problem into a dynamical one* with the transformation

$$f \rightarrow v \text{ (nonlinear)}, \quad (29)$$

and

$$x \rightarrow t \text{ (linear)}. \quad (30)$$

Our second result reads: *the choice of q determines just what type of dynamical problem is to be addressed.* Examples of such problems follow.

IV. RELATIVISTIC AND MECHANICAL APPLICATIONS

In this section we stress the connection between the probability distribution and the cosmological expansion factor, for a closed Friedman-Robertson-Walker spacetimes. Also, we list a set of simple mechanical problem connected with a non-extensive statistical setting.

A. Cosmological scenario

Friedman-Robertson-Walker spacetimes (FRW) are specially important in providing cosmological models which are in good agreement with observation. In this section we apply the equations of gravitation to a space which over its whole extent is completely homogeneous and isotropic. The metric is given by

$$ds^2 = -dt^2 + a^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (31)$$

where a/\sqrt{k} is the “radius of curvature” of the space and k is the curvature of the space. This metric allows for a particular calculational simplicity, on account of the high degree of symmetry and the single metric degree of freedom. Also, we are dealing with a geometry not very different from that of the actual universe, in its later stages.

In order to give a good representation of the scale factor a as a probability density, we choose an space with positive curvature that means $k > 0$. In this case, another convenient form for the four-dimensional spherical coordinates is obtained by introducing in place of the coordinates r the angle α according to $r = \frac{1}{\sqrt{k}} \sin \alpha$ (α goes between the limits 0 to 2π). Then

$$ds^2 = -dt^2 + \frac{a^2}{k} [d\alpha^2 + \sin^2 \alpha (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (32)$$

The coordinate α determines the distance from the origin, given by $\alpha a/\sqrt{k}$. The surface of a sphere in these coordinates equals $\frac{4\pi a^2}{k} \sin^2 \alpha$. We see that as we move away from the origin, the surface of the sphere increases, reaching its maximum value $4\pi a^2/k$ at a distance of $\pi a/2\sqrt{k}$. After that it begins to decrease, and it reduces itself to a point at the opposite pole of the space, at a distance $\pi a/\sqrt{k}$, the largest distance which can in general exist in such a space if we note that the coordinate r cannot take on values greater than $1/\sqrt{k}$.

The gravitational action $S[g]$ can be written as

$$S[g] = S_E[g] + \int \sqrt{-g} \mathcal{L}_m d^4x, \quad (33)$$

where S_E is the classical Einstein gravitational action, \mathcal{L}_m is the matter Lagrangian and g denotes the metric determinant. Normally the classical Einstein gravitational action S_E is

$$S_E[g] = \frac{1}{16\pi G} \int \sqrt{-g} R d^4x + (\text{surface terms}), \quad (34)$$

where G is the gravitational constant, R the scalar curvature and the surface terms (st) are to be chosen so as to cancel those arising from the integration by parts of the second derivatives terms contained in R . Throughout, we use units that entail $c = 1$. The variational problem in Eq.(33) leads to Einstein's equation for the classical geometry

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi G T_{ik}, \quad (35)$$

where T_{ik} is the energy-momentum tensor of matter and fields, and R_{ik} is the Ricci tensor with $i, k = 1, \dots, 4$.

For geometries of the form Eq.(32) the scalar curvature is given by

$$R = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right], \quad (36)$$

where the dot denotes derivative with respect to t . We see that for these cases the gravitational action Eq.(34) acquires the appearance

$$S[g] = \frac{1}{16\pi G k^{3/2}} \int_0^{2\pi} \int_0^\pi \int_0^\pi \int_{t_1}^{t_2} a^3 \sin^2 \alpha \sin \theta R d\alpha d\theta d\phi dt + (\text{st}), \quad (37)$$

inserting the scalar curvature (36) into the gravitational action (33), we get

$$S[g] = \frac{3\pi}{4Gk^{3/2}} \int_{t_1}^{t_2} (-a\dot{a}^2 + ka) dt + \frac{3\pi}{4Gk^{3/2}} [a^2\dot{a}]_{t_1}^{t_2} + (\text{st}). \quad (38)$$

Taking into account that the surface terms (st) can be chosen to cancel those arising from the integration by parts of the second derivatives terms contained in (36) the last three terms in (11) can be cancelled. Finally

$$S[g] = -\frac{3\pi}{4Gk^{3/2}} \int_{t_1}^{t_2} (a\dot{a}^2 - ka) dt. \quad (39)$$

Below we shall compare the gravitational action (39) with the generalized action that arises from a minimization of Fisher's generalized information I_q as constrained by J (12) and (13). It is to be mentioned that EPI has been used to derive the Einstein field equations of gravitation, in a manner quite different from the one here discussed, in Ref. [13], where the ordinary ($q = 1$) I is employed. Here we are not directly using EPI, but a variational procedure that involves our Fisher quantity I_q .

From the identification of equations (12), (14), (15), (25), (27) and (28) with (39), we obtain

$$S[g] = -\frac{3\pi}{4Gk^{3/2}} \mathcal{A}_q = -\frac{3\pi}{4Gk^{3/2}} (I_3 - J), \quad (40)$$

for

$$q = 3$$

with

$$a = f, \quad t = x, \quad k = \lambda, \quad (41)$$

and

$$\frac{4}{9} \dot{v}^2 + k v^{\frac{2}{3}} = E, \quad (42)$$

where $E > 0$ in order to obtain a real probability distribution f and the effective force $F_{eff} = -2k/3v^{1/3}$ is attractive. At this stage we have reduced the gravitational closed (FRW) problem to a simple analogous mechanical problem, which in turn, is equivalent to that posed by a non-extensive statistical scenario with $q = 3$ [27].

The comparison $a(t) = f(x)$ (from (41)) shows that *the probability distribution can be identified with the radius of curvature of the space by choosing the cosmological time variable t proportional to the information parameter x , while the curvature of the space k is identified with λ , the normalization constant.* In this gravitational problem the variable $v = a^{3/2}$ is a function of the radius of curvature and Eq. (42) gives the dynamics for the Robertson-Walker metric when the right hand side of (42) is the energy density of the matter.

To see that, let us write now the 00 component of Einstein equation with no cosmological constant for the Robertson-Walker metric in the case of a perfect fluid source, satisfying an equation of state $p = (\sigma - 1)\rho$:

$$3H^2 + 3\frac{k}{a^2} = 8\pi G \frac{\rho_0}{a^{3\sigma}}, \quad (43)$$

where $H = \dot{a}/a$ is the expansion rate and ρ_0, σ are constants. Identifying the energy E with the constant $8\pi G/3$, Eq. (43) transforms into Eq.(42), with $f(x) = a(t)$ and $\sigma = 1$. Consequently, we have a universe filled with a classical matter source. In other words, we have shown the equivalence between the equation of motion for the matter dominated universe and that equation that arises in dealing with a $q = 3$ non-extensive statistical scenario.

We thus are led to the non-linear differential equation (42) that should yield our "optimal" probability distribution f . From (42) it is easy to show that the probability distribution $f = v^{2/3} < E/k$, i.e., it has an upper positive limit and it is bounded. We conclude that this result has a physical meaning. Indeed, inserting $f(x_m) = E/k$ into (42), we obtain $\dot{f} = 0$. Thus, at x_m the probability distribution f has an extremum. On the other hand, inserting these results into (22) we have $\ddot{f} = -\frac{k^2}{2E}$ (negatively defined). An f -maximum at x_m ensues. According to the identifications $x \rightarrow t$ and $a(t) \rightarrow f(x)$, we conclude that the radius of curvature a/\sqrt{k} reaches a maximum at $x = x_m$ (i.e., at a finite time t_m). A contraction process takes place afterwards.

B. Mechanical applications

An interesting set of physical mechanical problem can be interpreted in a non-extensive statistical setting when the transformation given by Eqs. (29) and (30) is applied. In the follows we list these problems and add some pertinent comments. Notice that in all cases no special physical meaning is to be attached to the statistical quantity f (other than it minimizes Fisher's information measure with some constraints). What does possess a physical meaning is the dynamical quantity v that arises after implementing the transformation (29).

- $q = 1$

We deal with the harmonic oscillator potential $V(v) = \lambda v^2$, where v is the position coordinate and 2λ the stiffness constant k .

- $q = 2$

We have a constant field with a linear potential $V(v) = \lambda v$.

- $q = -1$

We face the centrifugal potential for a free particle in cylindrical coordinates $V(v) = \lambda v^{-2}$, with v the radial coordinate and 2λ the square of the angular momentum.

- $q = 1/2$

Calling $v = \varphi$ we have $V(v) = \lambda \varphi^4$, i.e., Higgs' potential for a massless scalar field, a crucial ingredient (in particle physics) for understanding spontaneous symmetry breaking.

- $q = -2$

We deal with a particle subjected to Coulomb's potential in spherical coordinates: $V(v) = \lambda v^{-1}$, with v the radial coordinate and λ a constant proportional to the electrical charge of the particle.

V. CONCLUSIONS

In the present communication we translated the rules of the Frieden-Soffer Information Transfer game into a NET parlance. By suitably playing the game we find, exact analytical solutions to special instances of classical dynamics. Two transformations

- from probability distribution f to dynamical variable v (nonlinear)
- from space x to time t (linear)

allow the demon to convert statistical into dynamical equations. In an orthodox $q = 1$ context, he needs to know the expectation value of the kinetic energy. In a variable q setting, he just needs to ascertain the normalization of the probability distribution.

Thus, dynamical solutions can be regarded as probability distributions, in a variable q NET-environment, that extremize Fisher's information measure for translation families. These solutions can then be added to the impressive collection elaborated by Frieden and Soffer [7].

That a NET environment may be the appropriate one in order to address simple dynamical problems seems to vindicate similar connections already reported, in a totally different (non-Fisher) context by other authors [25, 26]. The present results should be viewed within such a background-landscape.

The extension to a non-extensive setting of the Frieden-Soffer principle of Extreme Physical Information (EPI) is seen here to lead to a rather intriguing result: The Frieden-Soffer demon can translate *dynamical frameworks* into *statistical ones* and vice versa. Two apparently disparate scenarios (deterministic and stochastic) are seen to emerge from the *same* Principle. This confirms results obtained in a quite different manner by Cocke and Frieden using the standard $q = 1$ scenario [13]. EPI ($q = 1$) can also derive both stochastic and deterministic laws, according to the nature of the phenomenon under study [12]. The NET setting adds the feature of a *direct* translation between deterministic and stochastic environments. Thus, we believe to have developed here an interesting generalization to the work of [7].

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 - [27] I_3 obeys a modified version of the Cramer-Rao bound (see Ref. [23] for details of this bound for $q \neq 1$).